Exercise 3

A spring with a mass of 2 kg has damping constant 14, and a force of 6 N is required to keep the spring stretched 0.5 m beyond its natural length. The spring is stretched 1 m beyond its natural length and then released with zero velocity. Find the position of the mass at any time t.

Solution

In order to determine the spring constant, use the fact that 6 N is needed to stretch the spring 0.5 m.

$$F = k(x - x_0)$$

$$6 \text{ N} = k(0.5 \text{ m})$$

$$k = 12 \, \frac{\text{N}}{\text{m}}$$

Apply Newton's second law to obtain the equation of motion.

$$\sum F = ma$$

Use the fact that acceleration is the second derivative of position $a = d^2x/dt^2$ and the fact that the spring force $F = -kx$ and the damping force $F = -c(dx/dt)$ are the only forces acting on the mass.

$$-c\frac{dx}{dt} - kx = m\frac{d^2x}{dt^2}$$

Bring all terms to the left side.

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0 \tag{1}$$

This is a linear homogeneous ODE, so its solutions are of the form $x = e^{rt}$.

$$x = e^{rt} \quad \rightarrow \quad \frac{dx}{dt} = re^{rt} \quad \rightarrow \quad \frac{d^2x}{dt^2} = r^2e^{rt}$$

Plug these formulas into equation (1).

$$m(r^2e^{rt}) + c(re^{rt}) + k(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$mr^2 + cr + k = 0$$

Solve for r.

$$r = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

$$r = \left\{ \frac{-c - \sqrt{c^2 - 4mk}}{2m}, \frac{-c + \sqrt{c^2 - 4mk}}{2m} \right\}$$

Two solutions to the ODE are

$$\exp\left(\frac{-c - \sqrt{c^2 - 4mk}}{2m}t\right) \quad \text{and} \quad \exp\left(\frac{-c + \sqrt{c^2 - 4mk}}{2m}t\right).$$

By the principle of superposition, then, the general solution to equation (1) is

$$x(t) = C_1 \exp\left(\frac{-c - \sqrt{c^2 - 4mk}}{2m}t\right) + C_2 \exp\left(\frac{-c + \sqrt{c^2 - 4mk}}{2m}t\right),$$

where C_1 and C_2 are arbitrary constants. Differentiate it with respect to t to get the velocity.

$$\frac{dx}{dt} = C_1 \frac{-c - \sqrt{c^2 - 4mk}}{2m} \exp\left(\frac{-c - \sqrt{c^2 - 4mk}}{2m}t\right) + C_2 \frac{-c + \sqrt{c^2 - 4mk}}{2m} \exp\left(\frac{-c + \sqrt{c^2 - 4mk}}{2m}t\right)$$

Apply the initial conditions, $x(0) = 1$ and $x'(0) = 0$, to determine C_1 and C_2.

$$x(0) = C_1 + C_2 = 1$$

$$\frac{dx}{dt}(0) = C_1 \frac{-c - \sqrt{c^2 - 4mk}}{2m} + C_2 \frac{-c + \sqrt{c^2 - 4mk}}{2m} = 0$$

Solving this system of equations yields

$$C_1 = \frac{1}{2}\left(1 - \frac{c}{\sqrt{c^2 - 4mk}}\right) \quad \text{and} \quad C_2 = \frac{1}{2}\left(1 + \frac{c}{\sqrt{c^2 - 4mk}}\right),$$

meaning the displacement from equilibrium is

$$x(t) = \frac{1}{2}\left(1 - \frac{c}{\sqrt{c^2 - 4mk}}\right) \exp\left(\frac{-c - \sqrt{c^2 - 4mk}}{2m}t\right) + \frac{1}{2}\left(1 + \frac{c}{\sqrt{c^2 - 4mk}}\right) \exp\left(\frac{-c + \sqrt{c^2 - 4mk}}{2m}t\right).$$

Therefore, plugging in $m = 2$ kg and $k = 12$ N/m and $c = 14$ N · s/m,

$$x(t) = -\frac{1}{5}e^{-6t} + \frac{6}{5}e^{-t}.$$

Below is a plot of $x(t)$ versus t.